HSC Mathematics Assessment Task 2 **March 2008**

Time Allowed: 65 minutes + 2 minutes reading time

Instructions: Start each question on a new page.
Show all necessary working, writing on one side of the paper only.
Work down the page and do not work in columns.
Marks may not be awarded for untidy or poorly arranged work.

Name:____

Class:____

Marks

H4	H5	H6	H7
	1		
		2	
3c	3ab		
	4b	4a	4c
	5	5	

Question 1 (11 marks)

(a) Evaluate $\sum_{k=2}^{6} k^3$ **2**

(b) Find the 100th term of the series $2+5+8+....$ **2**

(c) In a geometric series the first term is –4.
Find the common ratio if the 4th term is –32. **2**

(d) The series $3+6(x-1)^2+12(x-1)^4+...$ has a common ratio of 20. Find x. **3**

(e) An arithmetic series has a common difference of d.
Find, in terms of d the first term of the series if the sum of the first five terms is zero. **2**

Question 2 (11 marks)

Consider the function $f(x)=x^3-2x^2$

(i) Find $f'(x)$ and $f''(x)$. **2**

(ii) Show that the graph of $y=f(x)$ has two stationary points and determine their nature. **4**

(iii) Show that the graph of $y=f(x)$ has one point of inflexion **2**

(iv) Sketch the graph of $y=f(x)$ showing all turning points, points of inflexion and intercepts. **3**

Question 3 (11 marks)

(a) Two dice are rolled. What is the probability that the total is greater than 3? **2**

(b) The probability that it will rain in Chicago on any day in November is $\frac{1}{3}$.

The probability that it will snow in Chicago on any day in November is $\frac{1}{5}$.

The probability that there will be both rain and snow in Chicago on any day in November is $\frac{1}{10}$.

(i) What is the probability of neither rain nor snow in Chicago on any day in November? **2**

(ii) If two days in November are selected at random, what is the probability that it will snow on exactly one of the days? **2**

(c) The volume of a closed cylinder with radius r cm and height h cm is $16\pi\ \text{cm}^3$

(i) Given that the formula for the surface area of a cylinder with radius r cm and height h cm is $A = 2\pi r(r+h)$, show that $A = 2\pi\left(r^2 + \frac{16}{r}\right)$ **1**

(ii) Find the minimum possible surface area of this cylinder. **4**

Question 4 (11 marks)

(a) Consider the function $f(x) = 8x^3 - 6x$.
Show that the graph of $y = f(x)$ is concave up when $x > 0$. **3**

(b) If $g''(x) = 3x$, $g(4) = 6$ and $g'(4) = -1$, find $g(x)$. **4**

(c) Find the possible values of k if $h(x) = -\frac{x}{3}(x^2 + 6x + 3k)$ is a decreasing function for all x. **4**

Question 5 (11 marks)

(a) The sum of the first 8 terms of an arithmetic series is -18 and the 99th term is 45.
Find the first term and the common difference of the series. **3**

(b) In a series the formula for the *n*th term is given by $T_n = 3 + 2n$.
Find a formula in terms of *n* for the sum $T_2 + T_3 + T_4 + T_5 + ... + T_n$. **4**

(c) In a series the sum of *n* terms is given by the formula $\dfrac{n(n^2 - 1)}{3}$.
Find the formula for the *n*th term of the series. **4**

End of paper.

Solutions to Mathematics Asssessment 2

Question 1

(a) $$\sum_{k=2}^{6} k^3 = 2^3 + 3^3 + 4^3 + 5^3 + 6^3$$
$$= 440$$

(b) $8 - 5 = 5 - 2 = 3$

The series is arithmetic with first term 2 and common difference 3

$$T_n = a + (n-1)d$$
$$T_{100} = 2 + (100-1)3$$
$$= 299$$

(c) $$T_n = ar^{n-1}$$
$$-32 = (-4)r^{4-1}$$
$$r^3 = 8$$
$$r = 2$$

(d) $$2\left(x-1\right)^2 = 20$$
$$\therefore \left(x-1\right)^2 = 10$$
$$\therefore x = \pm\sqrt{10} + 1$$

(e) $$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$
$$S_5 = \frac{5}{2}\left(2(a) + (5-1)d\right)$$
$$0 = 10a + 20d$$
$$a = -2d$$

Question 2

(a) (i) $f'(x) = 3x^2 - 4x$

$f''(x) = 6x - 4$

(ii) Stationary points occur when $f'(x) = 0$

$\therefore x(3x-4) = 0$

$\therefore x = 0, \frac{4}{3}$

$$\begin{aligned} f(0) &= 0^3 - 2(0)^2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f\left(\frac{4}{3}\right) &= \left(\frac{4}{3}\right)^3 - 2\left(\frac{4}{3}\right)^2 \\ &= -\frac{32}{27} \end{aligned}$$

$$\begin{aligned} f''(0) &= 6(0) - 4 \\ &= -4 \\ &< 0 \end{aligned}$$

$$\begin{aligned} f''\left(\frac{4}{3}\right) &= 6\left(\frac{4}{3}\right) - 4 \\ &= 4 \\ &> 0 \end{aligned}$$

So there is a minimum turning point at $\left(\frac{4}{3}, -\frac{32}{27}\right)$ and a maximum turning point at $(0,0)$.

(iii) $f''(x) = 0$

$6x - 4 = 0$

$x = \frac{2}{3}$

x	0	$\frac{2}{3}$	1
$f''(x)$	-4	0	2

The second derivative is equal to zero and changes sign only at $x = \frac{2}{3}$ so there is one point of inflexion. (This point of inflexion is non-horizontal because $f'\left(\frac{2}{3}\right) \neq 0$)

(iv)

Question 3

(a) The probability the total is not greater than three is $\frac{3}{36}$

So the probability that the total is greater than three is $\frac{33}{36}$ i.e. $\frac{11}{12}$

(b) (i)

$$P(A \cup B) = P(A) + P(B) + P(A \cap B)$$
$$= \frac{1}{5} + \frac{1}{3} - \frac{1}{10}$$
$$= \frac{13}{30}$$

So $P(\overline{A \cup B}) = 1 - \frac{13}{30}$

$$= \frac{17}{30}$$

(ii)

$$P\left(S\overline{S}\right) + P\left(\overline{S}S\right) = \frac{1}{5} \times \frac{4}{5} + \frac{4}{5} \times \frac{1}{5}$$
$$= \frac{8}{25}$$

(c) (i) $\pi r^2 h = 16\pi$

$$h = \frac{16}{r^2}$$

$$A = 2\pi r(r+h)$$

$$= 2\pi r\left(r + \frac{16}{r^2}\right)$$

$$= 2\pi\left(r^2 + \frac{16}{r}\right)$$

(ii) Let $y = \left(r^2 + \frac{16}{r}\right) = r^2 + 16r^{-1}$

$$\frac{dy}{dr} = 2r - 16r^{-2}$$

$= 0$ at stationary points

$$\therefore 2r - \frac{16}{r^2} = 0$$

$$2r = \frac{16}{r^2}$$

$$r^3 = 8$$

$$r = 2$$

$$\frac{d^2y}{dx^2} = 2 + 32r^{-3}$$

$= 6$ when $r = 2$

> 0

So the minimum value of y occurs when $r = 2$

Since $A = 2\pi y$ and $2\pi > 0$ the minimum value of A occurs when $r = 2$

$$A = 2\pi\left(2^2 + \frac{16}{2}\right)$$

$$= 24\pi$$

The minimum surface area is 24π square cm.

Question 4 (12 marks)

(a) $f'(x) = 24x^2 - 6$

$f''(x) = 48x$

When $x > 0$, $48x > 0$ so that graph of $y = f(x)$ is concave up.

(b) $g'(x) = \frac{3x^2}{2} + C_1$

Now $\frac{3(4)^2}{2} + C_1 = -1$

$\therefore C_1 = -25$

$\therefore g'(x) = \frac{3x^2}{2} - 25$

$\therefore g(x) = \frac{x^3}{2} - 25x + C_2$

Now $6 = \frac{4^3}{2} - 25(4) + C_2$

$\therefore C_2 = 74$

$\therefore g(x) = \frac{x^3}{2} - 25x + 74$

(c) $h'(x) = -\left(x^2 + 4x + k\right)$

For $h(x)$ to be a decreasing function, $h'(x)$ must be negative for all x. I.e. $x^2 + 4x + k$ must be positive for all x.

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= 4^2 - 4(1)(k)\end{aligned}$$

$16 - 4k < 0$

$k > 4$

Question 5

(a) $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$

$\frac{8}{2}\left(2a + (8-1)d\right) = -18$

$16a + 56d = -36 \longleftarrow (1)$

$T_n = a + (n-1)d$

$a + (99-1)d = 45$

$a + 98d = 45 \longleftarrow (2)$

From (2) $a = 45 - 98d$
Into (1)
$16(45 - 98d) + 56d = -36$

$d = 0.5$

$a = 45 - 98(0.5)$

$a = -4$

The first term is -4 and the common difference is 0.5

(b) $T_n = 3 + 2n$

$= 3 + 2(n-1) + 2$

$= 5 + (n-1)2$

This has the form $a + (n-1)d$ so the series is arithmetic with first term 5 and common difference 2.

$$
\begin{aligned}
T_2 + T_2 + T_4 + T_5 + \ldots + T_n &= S_n - a \\
&= \frac{n}{2}\left(2a + (n-1)d\right) - a \\
&= \frac{n}{2}\left(2(5) + (n-1)(2)\right) - 5 \\
&= 5n + n(n-1) - 5 \\
&= n^2 + 4n - 5
\end{aligned}
$$

(c)
$$
\begin{aligned}
T_n &= S_n - S_{n-1} \\
&= \frac{n(n^2-1)}{3} - \frac{(n-1)\left((n-1)^2 - 1\right)}{3} \\
&= \frac{n(n+1)(n-1)}{3} - \frac{(n-1)(n^2-2n)}{3} \\
&= \frac{n(n+1)(n-1)}{3} - \frac{(n-1)n(n-2)}{3} \\
&= \frac{n(n-1)\left(n+1-(n-2)\right)}{3} \\
&= \frac{n(n-1)2}{3} \\
&= n(n-1)
\end{aligned}
$$

Checking

$$
\begin{aligned}
T_1 &= S_1 \\
&= \frac{1(1^2 - 1)}{3} \\
&= 1(1 - 1)
\end{aligned}
$$

So $S_n = n(n-1)$